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Dipole Čerenkov effect in doubly anisotropic media†

G P Sastry and S Bhattacharya

Department of Physics, Indian Institute of Technology, Kharagpur, India 721302

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Abstract. The combined Čerenkov output of a particle with an electric charge and a magnetic dipole moment moving uniformly in a generalized uniaxial medium is investigated in the rest frame of the particle. It is shown that the cross terms in the output add up to zero, so that the radiated powers of the electric and magnetic poles are simply additive. The integrals for the radiated energy are explicitly evaluated for an electric dipole in a singly anisotropic medium. From the results of this integration, the Čerenkov output of a magnetic dipole in a doubly anisotropic medium is obtained by making two sets of substitutions in succession.

1. Introduction

It is well known (Balazs 1956, Jelley 1958) that the dipole Čerenkov effect in isotropic media is too weak to be of practical value. The possibility of electromagnetic wave amplification in the presence of carrier drift (Bok and Nozieres 1963) has recently led to the preparation of new mixtures of ferromagnetic semi-metals (Veselago and Rudashevskii 1967). In view of their simultaneously large values of permittivity and permeability, these could form the basis of new types of Čerenkov magnetic moment detectors (Day 1961). However, the classical electrodynamics of such doubly anisotropic media has so far received relatively little attention (Lewandowski 1971, Sastry 1974).

In this paper, we evaluate the Čerenkov output of a particle with a charge and magnetic moment moving uniformly in a generalized uniaxial medium. The results are obtained by working in the rest frame of the particle and applying successively the reduction (Majumdar 1973) and duality (Papas 1965) theorems.

2. Duality substitutions in a magneto-electric medium

Viewed from the rest frame Σ of the particle, the moving medium appears to acquire a magneto-electric character. In this section, we set up the duality scheme (table 1) in a general magneto-electric medium.

With the formal introduction of the magnetic current density $K^i = (\boldsymbol{K}, \sigma)$, Maxwell's equations ($1a$, b) become symmetric and transform into one another under the substitu-

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Table 1. Duality scheme in a magneto-electric medium.

Maxwell's equations	$\partial_j H^{ij} = J^i, \quad \partial_j F^{*ij} = -K^i$	(1*a*, *b*)
Constitutive relations	$H^{ij} = \frac{1}{2} T^{ijkl} F_{kl}, \quad F^{*ij} = \frac{1}{2} B^{ijkl} H^*_{kl}$	(2*a*, *b*)
	$\begin{vmatrix} \boldsymbol{D} \\ \boldsymbol{H} \end{vmatrix} = \begin{vmatrix} \epsilon & \zeta \\ \xi & \lambda \end{vmatrix} \begin{vmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{vmatrix}, \quad \begin{vmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{vmatrix} = \begin{vmatrix} \alpha & \Sigma \\ \nu & \delta \end{vmatrix} \begin{vmatrix} \boldsymbol{D} \\ \boldsymbol{H} \end{vmatrix}$	(3*a*, *b*)
	$\alpha = (1 - \epsilon^{-1}\zeta\lambda^{-1}\xi)^{-1}\epsilon^{-1}$	
	$\delta = (1 - \lambda^{-1}\xi\epsilon^{-1}\zeta)^{-1}\lambda^{-1}$	
	$\Sigma = -(1 - \epsilon^{-1}\zeta\lambda^{-1}\xi)^{-1}\epsilon^{-1}\zeta\lambda^{-1}$	
	$\nu = -(1 - \lambda^{-1}\xi\epsilon^{-1}\zeta)^{-1}\lambda^{-1}\xi\epsilon^{-1}$	
Duality prescriptions	$H \to F^*, \quad J \to -K; \qquad F \to H^*, \quad K \to J$	(4*a*, *b*)
	$T^{ijkl} \to B^{ijkl}$	(4*c*)
	$\epsilon \to \delta, \quad \lambda \to \alpha, \quad \zeta \to -\nu, \quad \xi \to -\Sigma$	(4*d*)

tions (4*a*, *b*). On making these substitutions in any set of equations derived from Maxwell's equations, we therefore obtain a 'dual set' which is equally valid and yields the solution of the 'dual' problem in which the electric and magnetic sources switch their places. However, if the medium were to remain the same, these substitutions call for the additional substitution (4*c*), which can be seen from equations (3*a*, *b*) to be equivalent to equation (4*d*).

3. The integrals for the radiated energy

Let us consider a particle with charge e and magnetic dipole moment M moving uniformly with a velocity above the Čerenkov threshold in a doubly anisotropic medium with parallel principal frames of ϵ and μ. We assume that the medium satisfies the generalized uniaxiality condition $\epsilon_2\lambda_2 = \epsilon_3\lambda_3$ (Majumdar and Pal 1970). Let the motion of the particle take place along the x_1 axis lying in the principal x_1x_2 plane of the medium, such that the optic axis makes an angle Ω with the direction of motion.

In the rest frame Σ of the particle, the fields can be derived from two scalar potentials ϕ and Ψ as $\boldsymbol{E} = -\nabla\phi$, $\boldsymbol{H} = -\nabla\Psi$. The energy W radiated per unit path length is equal to the retarding force with which the fields react on the particle (Landau and Lifshitz 1960). Since the particle is at rest in Σ, the electric monopole experiences a force due only to the electric field while the magnetic dipole experiences a force due only to the magnetic field. However, since the moving medium is effectively magneto-electric, the electric monopole, though at rest, gives rise to a magnetic field and vice versa. W thus splits up into four parts:

$$W = W^{ee} + W^{eM} + W^{Me} + W^{MM}, \tag{5}$$

where W^{ee} is the force on the electric monopole due to the electric field of the electric monopole, W^{eM} is the force on the electric monopole due to the electric field of the magnetic dipole, etc.

The electric scalar potential produced by a magnetic monopole of moment m is given by (Majumdar and Pal 1970)

$$\phi^{em} = -\frac{m\beta\gamma^2}{8\pi^3}\int(\epsilon_1\lambda_1-\epsilon_2\lambda_2)\sin\Omega \times\left(\gamma(\lambda_2-\beta^2\epsilon_3)\cos\Omega\iint\frac{k_1k_3\,e^{ik.x}}{L(k)M(k)}\,d^3k-\lambda_2\sin\Omega\iint\frac{k_2k_3\,e^{ik.x}}{L(k)M(k)}\,d^3k\right), \quad (6)$$

where

$$L(k)=\gamma^2(\lambda_{22}-\beta^2\epsilon_3)k_1^2+\lambda_{11}k_2^2+\frac{\lambda_1\lambda_2}{\lambda_3}k_3^2-2\gamma\lambda_{12}k_1k_2, \quad (7a)$$

and

$$M(k)=\gamma^2(\epsilon_{11}\lambda_3-\beta^2\epsilon_1\epsilon_2)k_1^2+\lambda_3\epsilon_{22}k_2^2+\lambda_3\epsilon_3k_3^2+2\gamma\lambda_3\epsilon_{12}k_1k_2. \quad (7b)$$

Now, by definition,

$$W^{em}=e(\partial_1\phi^{em})_{x\to 0}. \quad (8)$$

From the duality scheme of § 2, the magnetic scalar potential of an electric monopole is obtained from (6) by the substitutions

$$\phi^{em}\to\Psi^{me}, \qquad m\to e, \qquad \epsilon_\alpha\to 1/\lambda_\alpha, \qquad \lambda_\alpha\to 1/\epsilon_\alpha. \quad (9)$$

Under these substitutions, it can easily be verified that

$$L(k)\to\frac{1}{\lambda_3\epsilon_1\epsilon_2}M(k), \qquad M(k)\to\frac{1}{\epsilon_3\lambda_1\lambda_2}L(k). \quad (10)$$

The force on a magnetic monopole due to the magnetic field of an electric monopole is then given by

$$W^{me}=m(\partial_1\Psi^{me})_{x\to 0}=-W^{em}. \quad (11)$$

Thus the cross terms W^{em} and W^{me} in the combined output of an electric and a magnetic monopole add up to zero. It can be shown from a more detailed examination of the integrals that this is generally true and therefore the Čerenkov yields of electric and magnetic monopoles are merely additive.

It only remains to evaluate W^{ee} and W^{MM} in equation (5). W^{ee} is the well known Čerenkov output of a charge in a uniaxial medium. In order to evaluate W^{MM}, we first calculate the force $W^{PP}_{\lambda=1}$ on an electric dipole in a magnetically isotropic medium, and apply the reduction and duality theorems in succession. It can easily be seen that

$$W^{PP}_{\lambda=1}=-P_\mu P_\nu\partial_\mu\partial_\nu\partial_1\left(\frac{\phi^{ee}}{e}\right)_{x\to 0,\lambda=1}, \quad (12)$$

where ϕ^{ee} is given by the Fourier integral

$$\phi^{ee}=\frac{e}{8\pi^3}\left(\gamma^2\int(\lambda_3-\beta^2\epsilon_{22})\iint\frac{e^{ik.x}}{M(k)}\,d^3k +\beta^2\gamma^2\int\frac{\lambda_2}{\lambda_3}\sin^2\Omega(\epsilon_1\lambda_1-\epsilon_2\lambda_2)\iint\frac{k_3^2\,e^{ik.x}}{L(k)M(k)}\,d^3k\right). \quad (13)$$

By a Lorentz transformation of the wavevector it follows that k_1 in Σ is proportional to the frequency ω in Σ^0. Hence, the integrals over k_1 cannot be performed without a detailed knowledge of the dispersive properties of the medium. The remaining integrations can be carried out by applying the residue theorem in the complex k_3 plane and converting the k_2 integral into that over a unit circle. The dipole output (12) is the sum of six terms W_1, $W_2, \ldots$ containing P_1^2, P_2^2, P_3^2, P_1P_2, P_2P_3, and P_1P_3 respectively. Each of the derivatives in (12) brings in corresponding powers of k_1, k_2 and k_3 into the numerator of (13), while leaving the denominator untouched. Each of these integrals consists of a principal part and a contribution from the poles. Both of these vanish if odd powers of k_3 occur in the numerator. Due to the presence of k_1k_2 in the denominator of (13), the principal part vanishes if the numerator contains either odd powers of k_1 or odd powers of k_2 after division by the highest power of k_1k_2. On the other hand, the contribution from the poles vanishes for even powers of k_1. This is because the imaginary parts of ϵ and μ are positive for positive $\omega(k_1)$ and negative for negative $\omega(k_1)$ (Landau and Lifshitz 1960). This makes the contour traverse around the poles in opposite senses for opposite signs of k_1.

From these considerations it can easily be verified that the cross terms in (12) containing P_1P_2, P_2P_3 and P_1P_3 are identically zero. This means once again that the Čerenkov yields of dipoles oriented along the x_1, x_2, x_3 axes do not interfere with one another and are merely additive. Further, it can be checked that the principal parts of these remaining integrals W_1, W_2, W_3 also vanish, leaving only the contributions from the ordinary and extraordinary poles, which are the zeros of $L(k)$ and $M(k)$ respectively.

4. Application of the reduction and duality theorems

The results of integration of the preceding section can at once be extended so as to apply to a doubly anisotropic medium by effecting the following substitutions in the order indicated (Majumdar 1973, Sastry 1974):

$$\beta^2\gamma^2k_1^2\epsilon_{\alpha\beta}\to\epsilon_{\alpha\beta},\qquad \cos\Omega\to\left(\frac{\lambda_1}{\lambda_{11}}\right)^{1/2}\cos\Omega,\qquad \sin\Omega\to\left(\frac{\lambda_2}{\lambda_{11}}\right)^{1/2}\sin\Omega,$$
$$k_1\to\left(\frac{|\lambda|}{\lambda_{11}}\right)^{1/2}k_1,\qquad \epsilon_\alpha\to\epsilon_\alpha\lambda_\alpha,\qquad \epsilon_{\alpha\beta}\to\beta^2\gamma^2k_1^2\epsilon_{\alpha\beta},\tag{14}$$

with an overall multiplication factor of $(\lambda_{11}/|\lambda|)^{1/2}$. From the resulting expressions, the output of a magnetic dipole can be obtained by the duality prescription,

$$P_\nu\to M_\nu,\qquad \epsilon_\alpha\to 1/\lambda_\alpha,\qquad \lambda_\alpha\to 1/\epsilon_\alpha.\tag{15}$$

The final expressions then take the form

$$W=W^{ee}+W^{MM}=W^{ee}+W_1+W_2+W_3,\tag{16}$$

where

$$W^{ee}=\frac{e^2}{4\pi c^2}\int\left[\left(\frac{\lambda_{11}}{|\lambda|}\right)^{1/2}-\frac{1}{\beta^2}\frac{1}{(\epsilon_3\epsilon_{22})^{1/2}}\right]\omega\,d\omega,\tag{17}$$

$$W_1=\frac{M_1^2}{4\pi\beta^2\gamma^2c^4}\int\left[(\epsilon_3\epsilon_{22})^{1/2}-\frac{1}{\beta^2}\left(\frac{|\lambda|}{\lambda_{11}}\right)^{1/2}\right]\omega^3\,d\omega,\tag{18}$$

$$
\begin{aligned}
W_2 = \frac{M_2^2}{8\pi c^4} \int \frac{\omega^3\,d\omega}{\sin^2\Omega} \Bigg[\!\!\Bigg[& \frac{\epsilon_{22}^2}{\epsilon_1\lambda_3}(\epsilon_3\epsilon_{22})^{1/2} + \cos^2\Omega \left[\frac{\epsilon_2\epsilon_{22}}{\epsilon_1\lambda_3}(\epsilon_3\epsilon_{22})^{1/2} - 2\frac{\epsilon_2\epsilon_{22}^2}{\epsilon_1\lambda_3}\left(\frac{\lambda_1\lambda_2}{\lambda_3\lambda_{11}}\right)^{1/2}\right] \\
& + \frac{1}{\beta^2}\left\{\left[\frac{\epsilon_{22}}{\epsilon_1}(\epsilon_3\epsilon_{22})^{1/2} - 3\frac{\epsilon_{22}^2}{\epsilon_1}\left(\frac{\epsilon_3\lambda_1}{\epsilon_2\lambda_{11}}\right)^{1/2}\right] + \cos^2\Omega\left[5\frac{\epsilon_2}{\epsilon_1}(\epsilon_3\epsilon_{22})^{1/2}\right.\right. \\
& \left.\left. + 5\frac{\epsilon_{22}^2\lambda_1}{\epsilon_1\lambda_{11}}\left(\frac{\lambda_1\lambda_2}{\lambda_3\lambda_{11}}\right)^{1/2} - 8\frac{\epsilon_2\epsilon_{22}}{\epsilon_1}\left(\frac{\lambda_1\lambda_2}{\lambda_3\lambda_{11}}\right)^{1/2}\right]\right\} + \frac{1}{\beta^4}\left\{\frac{\epsilon_{22}^2\lambda_1\lambda_2}{\epsilon_1\epsilon_2\lambda_{11}}\left(\frac{\epsilon_2\lambda_1}{\epsilon_3\lambda_{11}}\right)^{1/2}\right. \\
& \left. + \cos^2\Omega\left[4\frac{\epsilon_{22}\lambda_1}{\epsilon_1\lambda_{11}}\left(\frac{|\lambda|}{\lambda_{11}}\right)^{1/2} - 2\frac{\epsilon_2}{\epsilon_1}\left(\frac{|\lambda|}{\lambda_{11}}\right)^{1/2} - 3\frac{\epsilon_{22}^2\lambda_1^2\lambda_2}{\epsilon_1\lambda_{11}^2}\left(\frac{\lambda_1}{\epsilon_2\epsilon_3\lambda_{11}}\right)^{1/2}\right]\right\}\Bigg]\!\!\Bigg],
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
W_3 = \frac{M_3^2}{8\pi c^4} \int \frac{\omega^3\,d\omega}{\sin^2\Omega} \Bigg[\!\!\Bigg[& \frac{\epsilon_{22}^2}{\epsilon_1\lambda_3}(\epsilon_3\epsilon_{22})^{1/2} + \cos^2\Omega \left[2\frac{\epsilon_3^2\epsilon_{22}}{\epsilon_1\lambda_2}\left(\frac{\lambda_3\lambda_{11}}{\lambda_1\lambda_2}\right)^{1/2} - 3\frac{\epsilon_2\epsilon_{22}}{\epsilon_1\lambda_3}(\epsilon_3\epsilon_{22})^{1/2}\right] \\
& + \frac{1}{\beta^2}\left\{\left[\frac{\epsilon_3\epsilon_{22}}{\epsilon_1}\left(\frac{\lambda_3\lambda_{11}}{\lambda_1\lambda_2}\right)^{1/2} - 3\frac{\epsilon_{22}}{\epsilon_1}(\epsilon_3\epsilon_{22})^{1/2}\right] + \cos^2\Omega\left[5\frac{\epsilon_2\epsilon_{22}}{\epsilon_1}\left(\frac{\lambda_1\lambda_2}{\lambda_3\lambda_{11}}\right)^{1/2}\right.\right. \\
& \left.\left. - 3\frac{\epsilon_2}{\epsilon_1}(\epsilon_3\epsilon_{22})^{1/2}\right]\right\} + \frac{1}{\beta^4}\frac{\epsilon_{22}\lambda_2}{\epsilon_1\lambda_{11}}\left(\frac{|\lambda|}{\lambda_{11}}\right)^{1/2}\sin^2\Omega\Bigg]\!\!\Bigg].
\end{aligned}
\tag{20}
$$

The geometry of the problem singles out a unique plane in space, namely that containing the optic axis and the direction of motion. Thus, the two orientations of the dipole at right angles to the direction of motion are not equivalent. This explains why W_2 and W_3 are not in general equal. However, in the special case $\Omega = 0$ when the two directions coincide, there is no longer a unique plane, and the difference between the two transverse directions disappears. This is borne out by the fact that in this case both W_2 and W_3 reduce to the same expression:

$$
W_2(W_3) = \frac{M_2^2(M_3^2)}{8\pi c^4} \int \frac{\epsilon_2\epsilon_3(\epsilon_2\epsilon_3)^{1/2}\mu_1}{\epsilon_1}\left(1 - \frac{1}{\beta^2\epsilon_2\mu_3}\right)^2 \omega^3\,d\omega. \tag{21}
$$

In an isotropic medium, this should always be true, as can easily be verified from equations (19) and (20).

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